Math 121 2.4-2nd The Quotient Rule

objectives

1) Find derivatives of $\frac{f(x)}{g(x)}$ using the Quotient Rule

2) Find and interpret
   marginal average cost
   marginal average revenue

* marginal = derivative
* average = divide by the # of units x.

The Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot f'(x) - g'(x)\cdot f(x)}{[g(x)]^2}$$

* CAUTION: Because the numerator is subtracted, it is essential that you construct it in the correct order, or you will always have a sign error.

① Find the derivative two ways
a) by simplifying first
b) by using the quotient rule

$$f(x) = \frac{x^{10}}{x^{20}}$$

a) $f(x) = \frac{x^{10}}{x^{20}} = x^{10-20} = x^{-10}$ exponent laws

$f'(x) = -10x^{-10-1} = \boxed{-10x^{-11}} = \boxed{\frac{-10}{x^{11}}}$ power rule

b) $f'(x) = \frac{x^{20}\cdot\frac{d}{dx}[x^{10}] - x^{10}\cdot\frac{d}{dx}[x^{20}]}{[x^{20}]^2}$ quotient rule

$= \frac{x^{20}\cdot 10x^{9} - x^{10}\cdot 20x^{19}}{x^{40}}$ power rule

$= \frac{10x^{29} - 20x^{29}}{x^{40}}$ exponent laws.

$= \frac{-10x^{29}}{x^{40}}$ combine like terms

$= \boxed{-10x^{-11}} = \boxed{\frac{-10}{x^{11}}}$ exponent laws.

Find the derivative.

(2) $g(x) = \dfrac{3x^4}{x^3-2}$

$$g'(x) = \frac{(x^3-2)\cdot\frac{d}{dx}(3x^4) - 3x^4\cdot\frac{d}{dx}(x^3-2)}{(x^3-2)^2}$$ quotient rule

$$= \frac{(x^3-2)(12x^3) - 3x^4(3x^2)}{(x^3-2)^2}$$ power rule

$$= \frac{12x^6 - 24x^3 - 9x^6}{(x^3-2)^2}$$ distribute, multiply

$$= \frac{3x^6 - 24x^3}{(x^3-2)^2}$$ combine like terms

$$= \boxed{\frac{3x^3(x^3-8)}{(x^3-2)^2}}$$

Even better... do you remember how to factor a difference of cubes $a^3-b^3 = (a-b)(a^2+ab+b^2)$

S same sign, O opposite sign, AP always positive

$$x^3 - 2^3 = (x-2)(x^2+2x+4)$$

$$= \boxed{\frac{3x^3(x-2)(x^2+2x+4)}{(x^3-2)^2}}$$

Because fractions are simplified by factoring, we leave final answers fully factored.

③ The cost of purifying a gallon of water to $x\%$ purity is $C(x) = \dfrac{2}{100-x}$ for $80 < x < 100$

where C is in dollars.
Find the rate of change of the purification costs when the purity is
a) 90%
b) 98%

Rate of change $\Rightarrow$ derivative!
(slope)

$$c'(x) = \frac{(100-x)\cdot \frac{d}{dx}(2) - 2\cdot\frac{d}{dx}(100-x)}{(100-x)^2}$$ quotient rule

$$= \frac{(100-x)\cdot 0 - 2(-1)}{(100-x)^2}$$ constant rule
power rule

$$c'(x) = \frac{2}{(100-x)^2}$$

a) $c'(90) = \dfrac{2}{(100-90)^2} = \dfrac{2}{10^2} = \dfrac{2}{100} = \dfrac{1}{50} = \boxed{\$.02/\text{gal}}$

b) $c'(98) = \dfrac{2}{(100-98)^2} = \dfrac{2}{2^2} = \dfrac{2}{4} = \dfrac{1}{2} = \boxed{\$.50/\text{gal}}$

Contrast:

$C(90) = \dfrac{2}{100-90} = \dfrac{2}{10} = \$.20$ to purify one gallon when purifying to 90%.

but <u>next</u> % better will cost $c'(90) = \$0.02$ <u>more</u>.

$c(98) = \dfrac{2}{100-98} = \dfrac{2}{2} = \$1.00$ to purify one gallon when purifying to 98%

but next % better will cost $c'(98) = \$0.50$ <u>more</u>.

Average cost per unit

$$AC(x) = \frac{C(x)}{x}$$ ← Total cost of producing x items

← divide by x items

$AC(x) =$ cost per item.

Notice: No calculus involved. This is like:

"It costs \$100 to produce 40 widgets.

$\uparrow$ $C(x)$ $\quad$ $\uparrow$ x

so it cost $\frac{\$100}{40} = \$2.50$ per widget."

Average Revenue per unit

$$AR(x) = \frac{R(x)}{x}$$ ← Total revenue of selling x items

← divide by x items.

"We made \$150 from selling 40 widgets.

$\uparrow$ $R(x)$ $\quad$ $\uparrow$ x

so our revenues are $\frac{\$150}{40} = \$3.75$ per widget."

Average Profit per unit

$$AP(x) = \frac{P(x)}{x}$$ ← Total profit (Revenue − Costs) of selling x items

← divide by x. items.

" Our profit from selling 40 widgets

$R(x) - C(x) = P(x)$

$150 - 100 = \$50$

was $\$50/40 = \$1.25$ per widget."

No calculus on this page; just three examples of per-unit calculations.

To create a per unit calculation, always divide by the # of objects.

MARGINAL means rate of change = slope = derivative.

marginal average cost = derivative of average cost

marginal average revenue = derivative of average revenue

marginal average profit = derivative of average profit.

* This book introduces profit and revenue, but seems to have only cost problems.

(4) A company can produce computer flash memory devices at a cost of \$6 each, while fixed costs are \$50 per day. Therefore the company's daily cost function is $C(x) = 6x + 50$.

a) Find the average cost function

b) Find the marginal average cost function.

a) $AC(x) = \frac{C(x)}{x}$ meaning of average cost

$\boxed{AC(x) = \frac{6x+50}{x}} = \frac{6x}{x} + \frac{50}{x} = \boxed{6 + 50x^{-1} = AC(x)}$

divide simplify.

b) $MAC(x) = \frac{d}{dx}\left[\frac{c(x)}{x}\right]$

$= \frac{d}{dx}(6 + 50x^{-1})$

$= 0 - 50x^{-2}$

$\boxed{MAC(x) = \frac{-50}{x^2}}$

Find derivative 3 ways

5. $f(x) = (x^5+1)\dfrac{x^3+2}{x+1}$

a) product of (x^5+1) with quotient $\left(\dfrac{x^3+2}{x+1}\right)$

b) quotient of the product $(x^5+1)(x^3+2)$ over $(x+1)$

c) simplify numerator and use only quotient.

a) $f'(x) = \dfrac{d}{dx}(x^5+1)\cdot\left[\dfrac{x^3+2}{x+1}\right] + (x^5+1)\cdot\dfrac{d}{dx}\left[\dfrac{x^3+2}{x+1}\right]$ product rule

$= (5x^4+0)\cdot\left[\dfrac{x^3+2}{x+1}\right] + (x^5+1)\cdot\left[\dfrac{(x+1)\cdot\frac{d}{dx}(x^3+2) - (x^3+2)\cdot\frac{d}{dx}(x+1)}{(x+1)^2}\right]$

power rule

quotient rule

$= 5x^4\left[\dfrac{x^3+2}{x+1}\right] + (x^5+1)\left[\dfrac{(x+1)\cdot 3x^2 - (x^3+2)\cdot 1}{(x+1)^2}\right]$ power rule

$= \dfrac{5x^7+10x^4}{(x+1)} + (x^5+1)\left[\dfrac{3x^3+3x^2-x^3-2}{(x+1)^2}\right]$ distribute

dist

$= \dfrac{(5x^7+10x^4)(x+1)}{(x+1)^2} + \dfrac{(x^5+1)(2x^3+3x^2-2)}{(x+1)^2}$

find common denom

multiply numerators

$= \dfrac{5x^8+5x^7+10x^5+10x^4 + 2x^8+3x^7-2x^5+2x^3+3x^2-2}{(x+1)^2}$

$= \boxed{\dfrac{7x^8+8x^7+8x^5+10x^4+2x^3+3x^2-2}{(x+1)^2}}$

b) $f'(x) = \dfrac{(x+1)\cdot\frac{d}{dx}\left[(x^5+1)(x^3+2)\right] - \left[(x^5+1)(x^3+2)\right]\cdot\frac{d}{dx}(x+1)}{(x+1)^2}$ quotient rule

$= \dfrac{(x+1)\cdot\left[\frac{d}{dx}(x^5+1)\cdot(x^3+2) + \frac{d}{dx}(x^3+2)\cdot(x^5+1)\right] - \left[(x^8+2x^5+x^3+2)\cdot 1\right]}{(x+1)^2}$

product rule

mult

power rule

$= \dfrac{(x+1)\left[(5x^4)(x^3+2) + (3x^2)(x^5+1)\right] - x^8-2x^5-x^3-2}{(x+1)^2}$

dist neg

$$= \frac{(x+1)\left[5x^7+10x^4+3x^7+3x^2\right]-x^8-2x^5-x^3-2}{(x+1)^2}$$ distribute

$$= \frac{(x+1)(8x^7+10x^4+3x^2)-x^8-2x^5-x^3-2}{(x+1)^2}$$ combine

$$= \frac{8x^8+10x^5+3x^3+8x^7+10x^4+3x^2-x^8-2x^5-x^3-2}{(x+1)^2}$$ multiply

$$= \boxed{\frac{7x^8+8x^7+8x^5+10x^4+2x^3+3x^2-2}{(x+1)^2}}$$

c) $$f(x) = \frac{(x^5+1)(x^3+2)}{(x+1)}$$

$$f(x) = \frac{x^8+2x^5+x^3+2}{x+1}$$ FOIL/mult (no calculus yet)

$$f'(x) = \frac{(x+1)\cdot\frac{d}{dx}(x^8+2x^5+x^3+2)-(x^8+2x^5+x^3+2)\cdot\frac{d}{dx}(x+1)}{(x+1)^2}$$

$$= \frac{(x+1)(8x^7+10x^4+3x^2)-(x^8+2x^5+x^3+2)\cdot 1}{(x+1)^2}$$

$$= \frac{8x^8+10x^5+3x^3+8x^7+10x^4+3x^2-x^8-2x^5-x^3-2}{(x+1)^2}$$

$$= \boxed{\frac{7x^8+8x^7+8x^5+10x^4+2x^3+3x^2-2}{(x+1)^2}}$$